[27/A-19] **SEAT No._______**

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Sardar Patel University

M.Sc. Semester III Examination

Friday, 21st April 2017; 14:00 to 17:00

Subject: Mathematics; Code: PS03EMTH08; Title of Paper: Group Theory

Maximum Marks: 70

Note: Notations are standard.

Q.1 To answer, write the correct **question number and option number only.** [8]

(a) For an element a of a group G, $a^{m+n} =$ _____.

(i) a^{mn} (ii) $(a^m)^n$ (iii) $a^n a^m$ (iv) none of the other three

(b) The inverse of the permutation (1 2 3) is _____.

(i) (1 2 3) (ii) (2 3 1) (iii) $(1\ \frac{1}{2}\ \frac{1}{3})$ (iv) (2 1 3)

(c) For a finite abelian subgroups H, K of a group G, $\mathrm{o}(HK)\mathrm{o}(H \cap K) =$ _____.

(i) $\mathrm{o}(H)\mathrm{o}(K)$ (ii) $\mathrm{o}(H)/\mathrm{o}(K)$ (iii) $\mathrm{o}(H) + \mathrm{o}(K)$ (iv) $\mathrm{o}(HK)$

(d) In S_3 the number of elements in the conjugacy class $C(2\ 3)$ is _____.

(i) 1 (ii) 2 (iii) 3 (iv) 6

(e) Total number of conjugacy classes in S_4 is _____.

(i) 2 (ii) 3 (iii) 4 (iv) 5

(f) A group of order _____ is simple.

(i) 3 (ii) 4 (iii) 6 (iv) 8

(g) A group of order _____ is abelian.

(i) 6 (ii) 8 (iii) 16 (iv) 25

(h) There are _____ nonisomorphic finite abelian groups of order 16.

(i) 1 (ii) 3 (iii) 5 (iv) 7

Q.2 Attempt any Seven. **(Start a new page.)** [14]

(a) Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, where $a, b, c, d \in \mathbb{Z}$. Show that $ad - bc = \pm 1$.

(b) Let G be a group. Show that every $g \in G$ has a unique inverse in G.

(c) Let G be a group of order 18. Show that there is $a \in G \setminus \{e\}$ such that $a^4 = e$.

(d) Let G be a finite group and $a \in G$. Show that $\mathrm{o}(a) \mid \mathrm{o}(G)$.

(e) Prove that the relation of conjugacy is a transitive relation on a group.

(f) Give one reason to conclude that a group of order 9 is abelian.

(g) Let A, B be subgroups of a group G. For $x, y \in G$ define $x \sim y$ if there exist $a \in A, b \in B$ such that $y = axb$. Show that $\sim$ is transitive.

(h) Let G be an abelian group and $\widehat{G}$ be the set of all homomorphisms from G to the group of all nonzero complex numbers. Show that $\widehat{G}$ is abelian.

(i) For $n \in \mathbb{N}$ define $\varphi : S_n \to \mathbb{Z}_2$ by $\varphi(\theta) = \begin{cases} 0 & \text{if } \theta \text{ is even} \\ 1 & \text{if } \theta \text{ is odd.} \end{cases}$ Show that φ is a homomorphism.

Q.3 **(Start a new page.)**

(*a*) Define a *group* and show that the set S_3 of all permutations on three symbols is a nonabelian group. [6]

(*b*) Let H, K be two subgroups of a group G. Show that HK is a subgroup of G if and only if $HK = KH$. [6]

OR

(*b*) State and prove Cayley's Theorem [6]

Q.4 **(Start a new page.)**

(*c*) Define *automorphism* of a group. If G is a group, then prove that the set of all automorphisms of G is also a group. [6]

(*d*) Let G be a finite group and $a \in G$. In usual notations, prove that $c_a = \mathrm{o}(G)/\mathrm{o}(N(a))$. [6]

OR

(*d*) If G is a group with $\mathrm{o}(G) = p^n$ for some prime p and $n \in \mathbb{N}$, then show that $Z(G) \neq \{e\}$. [6]

Q.5 **(Start a new page.)**

(*e*) Let G be a finite group and $p \in \mathbb{N}$ be prime such that $p^n \mid \mathrm{o}(G)$. Show that G has a subgroup of order p^n. [6]

(*f*) Let $p \in \mathbb{N}$ be prime. Let $n(k) \in \mathbb{N}$ be such that $p^{n(k)} \mid (p^k)!$ but $p^{n(k)+1} \nmid (p^k)!$. Show that [6]

$$n(k) = 1 + p + p^2 + \cdots + p^{k-1}.$$

OR

(*f*) Define *solvable group*. Show that S_5 is not solvable. [6]

Q.6 **(Start a new page.)**

(*g*) Define *internal direct product*. Let G be an internal direct product of its subgroups $N_1, N_2, \ldots, N_k$ and let $1 \leq i < j \leq k$. Show that $N_i \cap N_j = \{e\}$. Also for $a \in N_i$, $b \in N_j$, show that $ab = ba$. [6]

(*h*) For to isomorphic abelian groups G and G', and $s \in \mathbb{N}$, show that $G(s)$ is isomorphic to $G'(s)$, where $G(s) = \{x \in G : x^s = e\}$. [6]

OR

(*h*) Let $p \in \mathbb{N}$ be a prime. Prove that the number of nonisomorphic abelian groups of order p^n is equal to the number of partitions of n. [6]

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