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Number of printed pages 2

[24/A-2]

**Sardar Patel University****M. Sc. (Second Semester) Examination****Tuesday, 25<sup>th</sup> October 2016****Course No. PS02CMTH04 : Functional Analysis – I****Time: 10.00 a.m. to 01.00 p.m.****Maximum marks: 70****Note:** Figures to the right indicates marks. K denotes the field R or C.

1. Fill up the gaps in the following from the given option : [8]

- i) Let H be a Hilbert space and  $x, y \in H$  be orthonormal. Then  $\|x + y\| = \underline{\hspace{2cm}}$ .  
 (a) 1 (b) 2 (c) 0 (d)  $\sqrt{2}$
- ii) For  $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$ , define  $\langle x, y \rangle = \underline{\hspace{2cm}}$ . Then  $\langle \cdot, \cdot \rangle$  is not an inner product on  $\mathbb{R}^2$ .  
 (a)  $x_1 y_1 + x_2 y_2$  (b)  $x_1 x_2 + y_1 y_2$   
 (c)  $5x_1 y_1 + 6x_2 y_2$  (d)  $6x_1 y_1 + 5x_2 y_2$
- iii) Let X be an inner product space and  $\{u_n\}$  be a sequence of orthonormal elements in X. Then  $\underline{\hspace{2cm}}$ .  
 (a)  $\|u_n\| \rightarrow 0$  (b)  $u_n \rightarrow 0$   
 (c)  $u_n \rightarrow 0$  weakly (d)  $\{u_n\}$  is Cauchy
- iv) Let X be an inner product space,  $E \subset X$  and  $x \in \bar{E}$ . If there is a best approximation form E to x, then  $\underline{\hspace{2cm}}$ .  
 (a)  $x = 0$  (b)  $x \in E^\circ$  (c)  $x \in E$  (d)  $x \in E$
- v) Let H be a Hilbert space and  $T \in BL(H)$  be such that  $\|Tx\| = \|x\|$  for each  $x \in H$ . Then  $\underline{\hspace{2cm}}$ .  
 (a) T is onto (b)  $T^*T = I$  (c)  $TT^* = I$  (d) T is not one to one
- vi) Let H be a Hilbert space and  $S, T \in BL(H)$  be self-adjoint. Then  $\underline{\hspace{2cm}}$  self-adjoint.  
 (a)  $3S + 4T$  (b)  $ST$  (c)  $3S + 4iT$  (d)  $3iS + 4T$
- vii) Let H be a Hilbert space and  $T \in BL(H)$ . Then  $\underline{\hspace{2cm}}$ .  
 (a)  $\ker(T^*) = \ker(T^*T)$  (b)  $\ker(T) = \ker(T^*T)$   
 (c)  $R(T) = R(T^*T)$  (d)  $R(T^*) = R(T^*T)$
- viii) Let H be a Hilbert space;  $T \in BL(H)$  be non-zero compact self-adjoint and  $0 \neq \lambda \in \sigma(T)$ . Then  $\underline{\hspace{2cm}}$ .  
 (a)  $\lambda \in \sigma_c(T)$  (b)  $T - \lambda I$  is onto  
 (c)  $T - \lambda I$  is one to one (d)  $T - \lambda I$  is invertible

2. Answer any SEVEN of the following: [14]

- i) Let H be a Hilbert space and  $T \in BL(H)$  be one to one. Define  $\langle \cdot, \cdot \rangle_T$  by  $\langle x, y \rangle_T = \langle Tx, Ty \rangle$ ,  $x, y \in H$ . Show that  $\langle \cdot, \cdot \rangle_T$  is an inner product on H.
- ii) State and prove Pythagoras Theorem in an Inner Product Space.
- iii) Show that every strongly convergent sequence in a Hilbert space is weakly convergent.
- iv) Calculate Gram matrix for  $x_1 = (4, 0, 0), x_2 = (0, 3, 2), x_3 = (4, 3, 2)$ .
- v) Define  $T : K^2 \rightarrow K^2$  by  $T(x(1), x(2)) = (x(1) + 2x(2), 3(1) + x(2)), (x(1), x(2)) \in K^2$ . Find  $T^*$ .
- vi) Let H be a Hilbert space and  $T \in BL(H)$ . Show that T is one to one iff  $R(T^*)$  is dense in H.
- vii) Let H be a Hilbert space and  $T \in BL(H)$  be bounded below. Show that there is a sequence  $\{x_n\}$  in H with  $\|x_n\| = 1$  for each n and  $Tx_n \rightarrow 0$ .
- viii) Let H be a Hilbert space and  $T \in BL(H)$  be normal. Show that  $\|T^*x\| = \|Tx\|$  for all  $x \in H$ .
- ix) Let H be a Hilbert space and  $S, T \in BL(H)$  be such that S is compact. Show that ST is compact.

3. a) State and prove Schwarz inequality in an inner product space. [6]  
 b) Let  $X$  be an infinite dimensional inner product space. If  $X$  is separable, prove that it has a countable orthonormal basis. [6]

**OR**

- b) Let  $X$  be an inner product space and  $E$  be an orthonormal subset of  $X$ . Show that for every  $x \in X$ , the set  $E_x = \{ u \in E : \langle u, x \rangle \neq 0 \}$  is countable. [6]  
 4. a) State and prove Riesz representation theorem. [6]  
 b) State and prove unique Hahn Banach extension theorem. [6]

**OR**

- b) Let  $H$  be a Hilbert space;  $E$  be a closed convex subset of  $H$  and  $x \in H$ . Show that there is a unique best approximation  $y$  from  $E$  to  $x$ . [6]  
 5. a) Let  $H$  be a Hilbert space and  $T \in BL(H)$  be bounded below. Show that  $R(T) = H$ . [6]  
 b) Define: unitary operator. Let  $H$  be Hilbert space;  $A, B \in BL(H)$  be self-adjoint. and  $T=A+iB$ . Show that  $T$  is unitary iff  $AB = BA$  and  $A^2 + B^2 = I$ . [6]

**OR**

- b) Let  $H$  be a Hilbert space and  $T \in BL(H)$ . Define the adjoint  $T^*$  of  $T$ . [6]  
 If  $S, T \in BL(H)$ , show that  $(ST)^* = T^*S^*$ ,  $(S+T)^* = S^*+T^*$  and  $\|T^*T\| = \|T\| = \|T^*\|$ . [6]  
 6. a) Let  $H$  be a Hilbert space and  $T \in BL(H)$ . Show that  $\sigma(T) = \sigma_a(T) \cup \{ \mu ; \bar{\mu} \in \sigma_e(T^*) \}$  [6]  
 b) Let  $H$  be a Hilbert space and  $\{T_n\}$  be a sequence of compact operators such that  $T_n \rightarrow T$  in  $BL(H)$ . Show that  $T$  is compact. [6]

**OR**

- b) Define Hilbert – Schmidt operator. Show that every Hilbert –Schmidt operator is compact. [6]

