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SEAT No.________

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SARDAR PATEL UNIVERSITY

M.Sc. (Mathematics) Semester - I Examination

Wednesday, 19^{th} April, 2017

PS01CMTH04, Linear Algebra

Time: 10:00 a.m. to 1:00 p.m. **Maximum marks: 70**

Note: All the questions are to be answered in answer book only. Figures to the right indicate full marks of the respective question. Assume standard notations wherever applicable.

Q-1 Fill up the gaps in the following: [8]

1. The dimension of the vector space $\mathbb{C}^2$ over $\mathbb{C}$ is _____.
   (a) 1 (b) 2 (c) 4 (d) infinite
2. Let V_1 and V_2 be two subspaces of a vector space V. Then _____ need not be a subspace of V.
   (a) $V_1 + V_2$ (b) $V_1 \cup V_2$ (c) $L(V_1 \cup V_2)$ (d) $V_1 \cap V_2$
3. Let V be a vector space over a field F and $T \in A(V)$ be such that $T^2 + 5T + I = 0$. Then _____.
   (a) T is regular (b) T is singular (c) $\det(T) = 0$ (d) T is not onto
4. Let V be a vector space over F and $T \in A(V)$ be such that $0 \neq T \neq I$ and $T^2 = T$. Then the characteristic roots of T are _____.
   (a) 0 and 1 (b) 0 and 2 (c) 0 and 0 (d) 1 and 1
5. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined as $T(x_1, x_2, x_3) = (0, x_1, x_2)$. Then the minimal polynomial of T is _____.
   (a) $p(x) = x$ (b) $p(x) = x^2$ (c) $p(x) = 0$ (d) $p(x) = x^3$
6. Let V vector space over F, W be subspaces of a V and $T \in A(V)$. W is invariant under T if _____.
   (a) T must be one-one (b) $T(W) = V$ (c) $T(W) \subset W$ (d) $W \subset T(W)$
7. Let $A \in M_n(F)$ be nilpotent. Then $\det(A) =$_____.
   (a) 1 (b) -1 (c) 0 (d) n
8. Let $A, B \in M_n(F)$ for some field F. Then _____.
   (a) $\text{tr}(\lambda A) = \lambda^n \text{tr}(A)$
   (b) $\det(\lambda A) = \lambda \det(A)$
   (c) $\det(A + B) = \det(A) + \det(B)$
   (d) $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$

Q-2 Attempt *Any Seven* of the following: [14]

(a) Check whether $v_1 = (1, 2, 1)$, $v_2 = (0, 1, 1)$ and $v_3 = (0, 0, 1)$ are linearly independent over $\mathbb{R}$ or not?

(b) Let $W = \{(x, y, z) \in \mathbb{R}^3 : 2x + 3y = z\} \subset \mathbb{R}^3$. Show that W is a subspace of $\mathbb{R}^3$. What is the dimension of W?

(c) Let V be a finite dimensional vector space over a field F and $T \in A(V)$. Show that $\text{rank}(ST) \leq \text{rank}(T)$.

(d) Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(x, y, z) = (x - y + z, x - 2y, x - 2z)$, $(x, y, z) \in \mathbb{R}^3$. Find the matrix of T with respect to standard basis of $\mathbb{R}^3$.

(e) Let V be a vector space over a field F and $T \in A(V)$. Show that $\ker(T)$ is invariant under T.

(f) Let V be a vector space over F and $S, T \in A(V)$ such that S is nilpotent. Show that ST is nilpotent if $ST = TS$.

(g) Show that similar matrices have same determinant.

(h) Find the inertia of the quadratic equation $x_1 + x_2 + x_3 = 0$.

(i) For $A, B \in M_n(\mathbb{R})$, show that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

Q-3 (a) Let V be a finite-dimensional vector space over a field F and W be a subspace of V. Show that W is finite-dimensional and $\dim V/W = \dim V - \dim W$. [6]

(b) Let V be a vector space and $\{v_1, v_2, \ldots, v_n\}$ be a basis of V. If $\{u_1, \ldots, u_m\}$ in V are linearly independent then $m \leq n$. [6]

OR

(b) Let V be a vector space over F. Show that V is isomorphic to a subspace of $\hat{\hat{V}}$. [6]

Q-4 (a) Let $\mathcal{A}$ be an algebra over F. Show that $\mathcal{A}$ is isomorphic to a subalgebra of $A(V)$ for some vector space V over F. [6]

(b) Let V be a vector space over F and $T \in A(V)$. Show that characteristic vectors corresponding to distinct characteristic roots of T are linearly independent. [6]

OR

(b) Let V be a vector space over F and $T \in A(V)$. Show that T is regular if and only if the constant term of the minimal polynomial for T is non-zero. [6]

Q-5 (a) Let V be a finite dimensional vector space over F and $T \in A(V)$ be nilpotent. Then show that the invariants of T are unique. [6]

(b) Let V be an n-dimensional vector space over F. If $T \in A(V)$ has all its characteristic roots in F, then show that T satisfies a polynomial of degree n in $F[x]$. [6]

OR

(b) Let V be a finite dimensional vector space over F and $T \in A(V)$. Let $V = V_1 \oplus V_2$, where V_1 and V_2 are subspaces of V invariant under T. Let $T_i = T|_{V_i}$ and $p_i(x) \in F[x]$ be the minimal polynomial for T_i, $i = 1, 2$. Show that the least common multiple of $p_1(x)$ and $p_2(x)$ is the minimal polynomial for T. [6]

Q-6 (a) For $A, B \in M_n(F)$, show that $\det(AB) = \det(A)\det(B)$. [6]

(b) i. Let V be a finite-dimensional vector space over F and $S, T \in A(V)$ such that $ST - TS$ commutes with S. Show that $ST - TS$ is nilpotent. [4]

ii. Find the symmetric matrix associated with the quadratic form: $-y^2 - 2z^2 + 4xy + 8xz - 14yz$. [2]

OR

(b) Prove that the determinant of a triangular matrix is the product of its entries on the main diagonal. [6]

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