SC

SEAT No.________

: 02

[74] Sardar Patel University, Department of Mathematics

M.Sc. (Mathematics) External Examination 2017;

Code:- PS01CMTH23 : Subject :- Functions of Several Real Variables;

Date: 03-11-2017, Friday; Time- 2.00 pm to 5.00 pm ; Max. Marks 70

Note: Notations and Terminologies are standard.

Q.1 Choose correct option from given four choices. [08]

(i) Let $x, y \in \mathbb{R}^n$ be orthogonal (i.e. perpendicular) vectors. Then $\|x+y\|^2 =$

(a) $\|x\|^2 + \|y\|^2$ (b) $\|x\| + \|y\|$ (c) $(\|x\| + \|y\|)^2$ (d) $\|x\|\|y\|$

(ii) Which of the following is true?

(a) $\lim\limits_{x\to 0} x\cos(\frac{1}{x}) = 0$ (b) $\lim\limits_{x\to 0} \frac{\sin x}{x} = 0$ (c) $\lim\limits_{x\to 0} \frac{\cos x}{x} = 1$ (d) none

(iii) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined as $f(t) = 3t^3$. Then $Df(2) =$

(a) λ_{12} (b) λ_{24} (c) λ_{36} (d) λ_{48}

(iv) Let $a = (2, 1)$ and $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined as $f(x) = x_1x_2$. Then $Df(a) =$

(a) $\pi_1 + \pi_2$ (b) $2\pi_1 + \pi_2$ (c) $\pi_1 + 2\pi_2$ (d) none

(v) Define $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ as $f(x) = 2x_1e^{x_2}$. Then $D_1f(0) =$

(a) -1 (b) 0 (c) 1 (d) 2

(vi) Let $a \in \mathbb{R}^n$ be fixed. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be continuous at a. Then

(a) $Df(a)$ exists (b) $D_xf(a)$ exist (c) $D_jf(a)$ exist (d) none

(vii) Let $S \in \mathcal{T}^3(V)$ and $T \in \mathcal{T}^5(V)$. Then $S \otimes T$ belongs to

(a) $\mathcal{T}^{15}(V)$ (b) $\mathcal{T}^8(V)$ (c) $\mathcal{T}^5(V)$ (d) $\mathcal{T}^2(V)$

(viii) The dimension of $\Lambda^6(\mathbb{R}^4)$ is

(a) 0 (b) 15 (c) 24 (d) 4096

Q.2 Attempt any seven. [14]

(i) Prove that $\|x\| \leq \sum_{i=1}^{n} |x_i|$ $(x \in \mathbb{R}^n)$.

(ii) Define oscillation $o(f; a)$ of $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ at a.

(iii) Define $f : \mathbb{R} \longrightarrow \mathbb{R}$ as $f(x) = x\cos(x)$. Find $Df(0) : \mathbb{R} \longrightarrow \mathbb{R}$.

(iv) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be differentiable at $a \in \mathbb{R}^n$. Then prove that its each component function $f^i : \mathbb{R}^n \longrightarrow \mathbb{R}$ is differentiable at a.

(v) If $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ has maximum value at a and $D_if(a)$ exists, then show that $D_if(a) = 0$.

(vi) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and $a, x \in \mathbb{R}^n$. Show that $D_{sx}f(a) = sD_xf(a)$ $(s \in \mathbb{R})$.

(vii) Define $T(x, y) = x_1y_2$ $(x, y \in \mathbb{R}^2)$. Find Alt(T).

(viii) Define tensor product and wedge product.

(ix) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be differentiable. Define $df(p)(v_p) := Df(p)(v)$ $(p \in \mathbb{R}^n;\ v_p \in \mathbb{R}^n_p)$. Prove that $df(p) : \mathbb{R}^n_p \longrightarrow \mathbb{R}$ is linear.

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Q.3

(a) Let $x, y \in \mathbb{R}^n$. Then Prove that $|\langle x, y\rangle| \leq \|x\|\|y\|$ and $\|x+y\| \leq \| x\| + \|y\|$. [6]

(b) Let $A \subset \mathbb{R}^n$, let $f : A \longrightarrow \mathbb{R}$ be a bounded function, and let $a \in A$. Then prove that f is continuous at a iff $o(f; a) = 0$. [6]

OR

(b) Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear map. Prove that there exists an $n \times m$ matrix A such that $T(x) = xA$ $(x \in \mathbb{R}^n)$. Further, if $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is defined as $T(x) = (x_1+x_3, x_1-2x_2+x_3)$, then find A corresponding to T. [6]

Q.4

(a) If a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is differentiable at $a \in \mathbb{R}^n$, then prove that there exists a unique linear transformation $\lambda : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that $\lim_{h\to 0} \frac{\|f(a+h)-f(a)-\lambda(h)\|}{\|h\|} = 0$. [6]

(b) Let $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}$ be differentiable at a. Prove that $f+g$ and fg are differentiable at a. [6]

OR

(b) Define $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{x_1|x_2|}{\|x\|} & (\text{if } x \neq 0) \\ 0 & (\text{if } x = 0) \end{cases}$$

Discuss the differentiability of f at 0. If it is differentiable at 0, then find its derivative. [6]

Q.5

(a) Let $a = (1, 0)$. Define $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ as $f(x) = (e^{x_1}, x_1 + \sin(x_2), \log(x_1) - x_2)$. Then find $f'(a)$ and $Df(a)$. [6]

(b) Prove that continuously differentiable function is differentiable. [6]

OR

(b) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be differentiable at a. Prove that $D_x f(a)$ exists for any $x \in \mathbb{R}^n$. Moreover, find $D_x f(0)$ for the function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ defined as $f(y) = 2y_1 + y_2^2$. [6]

Q.6

(a) Define $Alt(T)$. Prove that if $T \in \mathcal{T}^k(V)$, then $Alt(T) \in \Lambda^k(V)$. [6]

(b) Let $\omega \in \Lambda^k(V)$ and $\eta \in \Lambda^l(V)$. Then prove that $\omega \wedge \eta = (-1)^{kl}(\eta \wedge \omega)$. [6]

OR

(b) Define "k-form" on $\mathbb{R}^n$. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be differentiable. Then prove that [6]

$$\widetilde{f}_{1*}(d\pi_i) = \sum_{j=1}^{n} D_j f^i \cdot d\pi_j \quad (1 \leq i \leq m).$$

THE END