SC

SEAT No.________

[81]

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Sardar Patel University
Mathematics
M.Sc. Semester I
Wednesday, 01 November 2017
2.00 p.m. to 5.00 p.m.
PS01CMTH21 - Complex Analysis I

Maximum Marks: 70

Q.1 Fill in the blanks. [8]

(1) Let $a \in \mathbb{C}$. Then the minimum value of $\{|z-a|+|z+a| : z \in \mathbb{C}\}$ is _____

(a) 0 (b) $|a|$ (c) $2a$ (d) None of these

(2) $\text{Arg}(i) + \text{Arg}(-i) =$ _____

(a) 0 (b) π (c) 2π (d) $\{2n\pi : n \in \mathbb{Z}\}$

(3) If v and V are harmonic conjugates of a harmonic function u on a domain D, then which of the following is not true?

(a) $v = V$ (b) $v_x + V_y = v_y + V_x$ (c) $v_x = V_x$ (d) $V_{xx} + V_{yy} = 0$

(4) The set of singularity of the function $\cot 2z$ is _____

(a) $\{n\pi : n \in \mathbb{Z}\}$ (b) $\{2n\pi : n \in \mathbb{Z}\}$ (c) $\{\frac{n\pi}{2} : n \in \mathbb{Z}\}$ (d) $\{\frac{n\pi i}{2} : n \in \mathbb{Z}\}$

(5) $\int_{|z|=1} \frac{z}{z-2} dz =$ _____

(a) 0 (b) $4\pi i$ (c) $2\pi i$ (d) $\frac{1}{\pi i}$

(6) Which of the following is a bounded function on $\mathbb{C}$?

(a) $\cos z$ (b) e^{-z} (c) e^{-z^2} (d) none of these

(7) The Taylor series of $\frac{1}{1+z^2}$ about 2 is valid in $N(2, R)$ if $R =$_____

(a) $\sqrt{5}$ (b) $\sqrt{7}$ (c) $\sqrt{11}$ (d) $\sqrt{13}$

(8) The point 0 is a pole of $\frac{\tan z}{z^2}$ of order _____

(a) 1 (b) 2 (c) 3 (d) 4

Q.2 Attempt any ***Seven.*** [14]

(a) If $0 \neq \alpha \in \mathbb{C}$ and $\gamma \in \mathbb{R}$, then show that $\overline{\alpha}z + \alpha\overline{z} + \gamma = 0$ represents a line.

(b) If $\lim_{z \to z_0} f(z) = w_0$ and $w_0 \neq 0$, then show that there is $\delta > 0$ such that $|f(z)| > 0$ whenever $0 < |z - z_0| < \delta$.

(c) Let $n \in \mathbb{N} \setminus \{1\}$. Find the sum of all complex numbers satisfying $z^n = 2$.

(d) If $z \in \mathbb{C}$, then show that $\cosh^2 z - \sinh^2 z = 1$.

(e) If f is an entire function, then show that $g(z) = \overline{f(\overline{z})}$ is an entire function.

(f) Let m and n be integers, and let $C : z(\theta) = e^{i\theta}$, $0 \leq \theta \leq 2\pi$. Evaluate $\int_C z^n \overline{z}^m dz$.

(g) Let f be the function $f(z) = e^z$ and R the rectangular region $[0, 1] \times [0, \pi]$. Find the points in R where $u(x, y) = \text{Re } f(z)$ reaches its maximum and minimum values.
(h) Find the Taylor series of $\frac{1}{z-2}$ about i.
(i) Find the inverse of a bilinear transformation $w(z) = \frac{2z+3}{3z+2}$.

Q.3

(a) Let $f = u + iv$ be defined in a neighbourhood of $z_0 = x_0 + iy_0$. If the functions u_x, u_y, v_x, v_y are continuous in a neighbourhood of (x_0, y_0) and if $u_x(x_0, y_0) = v_y(x_0, y_0)$ and $u_y(x_0, y_0) = -v_x(x_0, y_0)$, then show that f is differentiable at z_0. [6]

(b) Define $\lim_{z \to z_0} f(z) = \infty$. If P is a polynomial of degree $n \geq 1$, then show that $\lim_{z \to \infty} P(z) = \infty$. [6]

OR

(b) Give an example of of a complex function which is differentiable at exactly one point. Show that the map $g \circ f$ is differentiable at z_0 whenever f is differentiable at z_0 and g is differentiable at $f(z_0)$. [6]

Q.4

(c) Let U be an open subset $\mathbb{C}$. Let $f : U \to \mathbb{C}$ be analytic such that $f'(z) = 0$ for all $z \in \mathbb{C}$. Can we conclude that f is a constant map? Why? If not, what condition on U implies that f is a constant map? Justify. [6]

(d) Let $N(z_0, R)$ be the disc of convergence of the power series $S(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$. If C is a contour in $N(z_0, R)$ and g is a continuous function C, then show that $\int_C g(z)S(z)dz = \sum_{n=0}^{\infty} a_n \int_C g(z)(z - z_0)^n dz$. State the results you use. [6]

OR

(d) Suppose that v is a harmonic conjugate of u on a domain D. Show that $f = u + iv$ is analytic on D. Find an analytic function f whose real part is $\frac{2xy}{(x^2+y^2)^2}$. [6]

Q.5

(e) If a function f is analytic and nonconstant in a domain D, then show that $|f|$ has no maximum value in D. State the results you use. [6]

(f) Let $C : z(t),\ a \leq t \leq b$, and let f be piecewise continuous on C. Define $\int_C f(z)dz$. If $f(z) = \pi \exp(\pi \bar{z})$ and C is the boundary of the square with vertices at the points 0, 1, $1 + i$ and i, the orientation of C being in the counterclockwise direction, then evaluate $\int_C f(z)dz$. [6]

OR

(f) If $v : \mathbb{R}^2 \to \mathbb{R}$ is a nonconstant harmonic function, then show that v is unbounded. State carefully the results you use. [6]

Q.6

(g) Let z_0 be an isolated singularity of f. Show that z_0 is a pole of f of order m if and only if there is a function φ which is analytic at z_0, $\varphi(z_0) \neq 0$ and $f(z) = \frac{1}{(z-z_0)^m}\varphi(z)$ for all z in some deleted neighborhood of z_0. Also, show that if $m = 1$, then $\text{Res}_{z=z_0} f = \lim_{z \to z_0}(z - z_0)f(z)$ and if $m > 1$, then $\text{Res}_{z=z_0} f = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}}[(z - z_0)^m f(z)]|_{z=z_0}$. [6]

(h) State Cauchy's Residue Theorem. Hence evaluate $\int_C \frac{\cot z}{z^4} dz$ and $\int_C \frac{\sinh z}{z^4(1-z^2)} dz$, where C is a positively oriented circle $|z| = \frac{1}{2}$. [6] [6]

OR

(h) State Laurent's Theorem. Find the Laurent series expansion of $\frac{1}{(z-1)(z-3)}$ about i in (all the three) appropriate regions. [6]

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